Mathematical Girls - Fermat's Last Theorem

C03

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YURI'S BEEN HOSPITALIZED?!
STAFF ROOM
SENPAI!!!
OKAY, I'LL DROP BY THE HOSPITAL TO SEE HER ON MY WAY HOME, THEN.
THEY SAY THERE WAS SOMETHING ABNORMAL WITH THE BONES IN HER LEG
I'M SORRY, I OVER-HEARD.
WOULD YOU MIND IF... WE GO SEE HER TOGETHER?

第3話
Chapter 3

Brighten
ONII-CHAN, YOU CAME TO SEE ME!!
OH!
TEHEHE
THAT'S SORT OF EMBARRASSING
BUT, HAVING THIS TURN INTO SOMETHING SERIOUS
I'M REALLY HAPPY, NYAA~
HOW'S YOUR CONDITION?
I'M FINE!
GLANCE

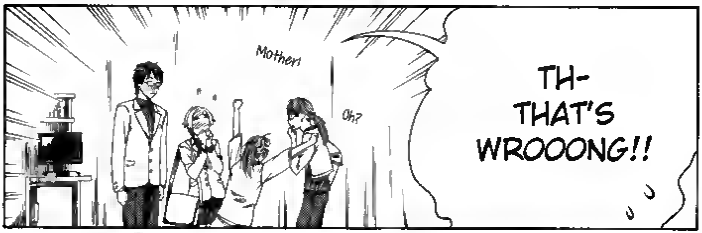
O.. ONII-CHAN, WHO IS THIS ?
MY KOUHAI, TETRA-CHAN, CAME TOGETHER WITH ME
TOGETHER?!
HERE, HAVE THIS
SMILE
NICE TO MEET YOU, YURI-CHAN

Please have a seat
AH
RATTLE
...SENPAI, SHE CALLED YOU ONII-CHAN?
YEAH,
YURI IS MY COUSIN, SHE HAS BEEN CALLING ME THAT SINCE LONG AGO...

EH!
ONII-CHAN!!
PATROLLING THE CLOCK THAT WE DID TOGETHER THE OTHER DAY WAS REALLY INTERESTING!!
WHEN THE NUMBER OF NUMBERS ON THE FACE OF THE CLOCK AND THE NUMBER OF STEPS WERE COPRIME, WE WERE ABLE TO GET COMPLETE PATROLS!
YURI LOVES WHEN ONII-CHAN TALKS ABOUT MATHEMATICS!!

ONII-CHAN IS NOW YURI'S FULL-TIME TEACHER!!
HEY ONII-CHAN!
I TOO, NPAI
SENPAI'S WAY OF TEACHING IS REALLY GOOD
FERMAT'S LAST THEOREM WAS INTERESTING TOO
IT WAS MARVELOUS THAT JUST BY TWISTING PYTHAGORAS' EQUATION A LITTLE BIT, WE END UP WITH NO SOLUTIONS IN THE NATURAL NUMBERS.

HOHOHO
YOU REALLY LOOK GOOD IN YOUR UNIFORM, DON'T YOU?
Poke out
OH~ YOU'RE BACK FROM SCHOOL?
M... MOTHER
SHUDDER
HMM!
HMMMMM!?



I'm a kouhai of his
My name is Tetra

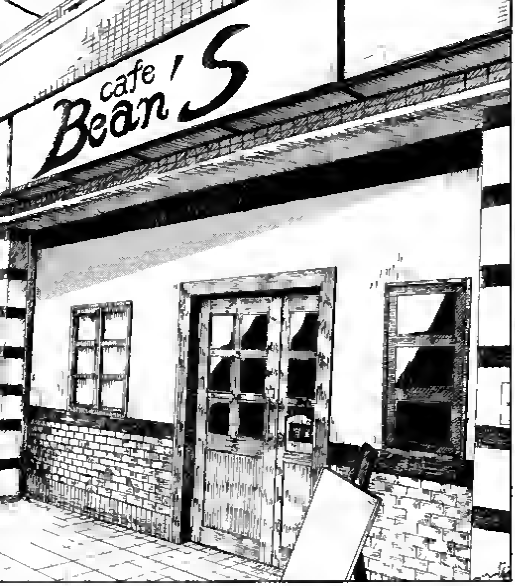
Oh my, you're
really polite

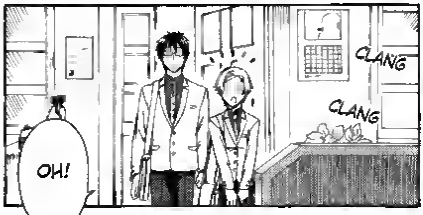
Well——

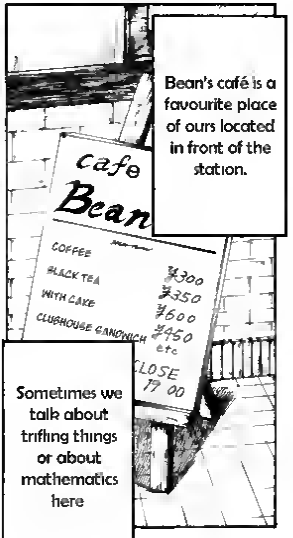
Hahaha.

I'M TIRED...
THAT'S BECAUSE YURI'S MOTHER IS REALLY TALKATIVE.
SIGH
YEAH.
BUT IT'S A RELIEF THAT YURI-CHAN SEEMS TO BE DOING WELL.

WAIT A MOMENT!
STEP STEP STEP
Ba-dump
OH! THERE YOU ARE.
YES...
EH?
MAY I ASK YOU TO COME?
SORRY,
BUT YURI SAYS THAT THERE'S SOMETHING SHE WANTS TO SAY TO YOUR GIRLFRIEND.
I'll be back in a while

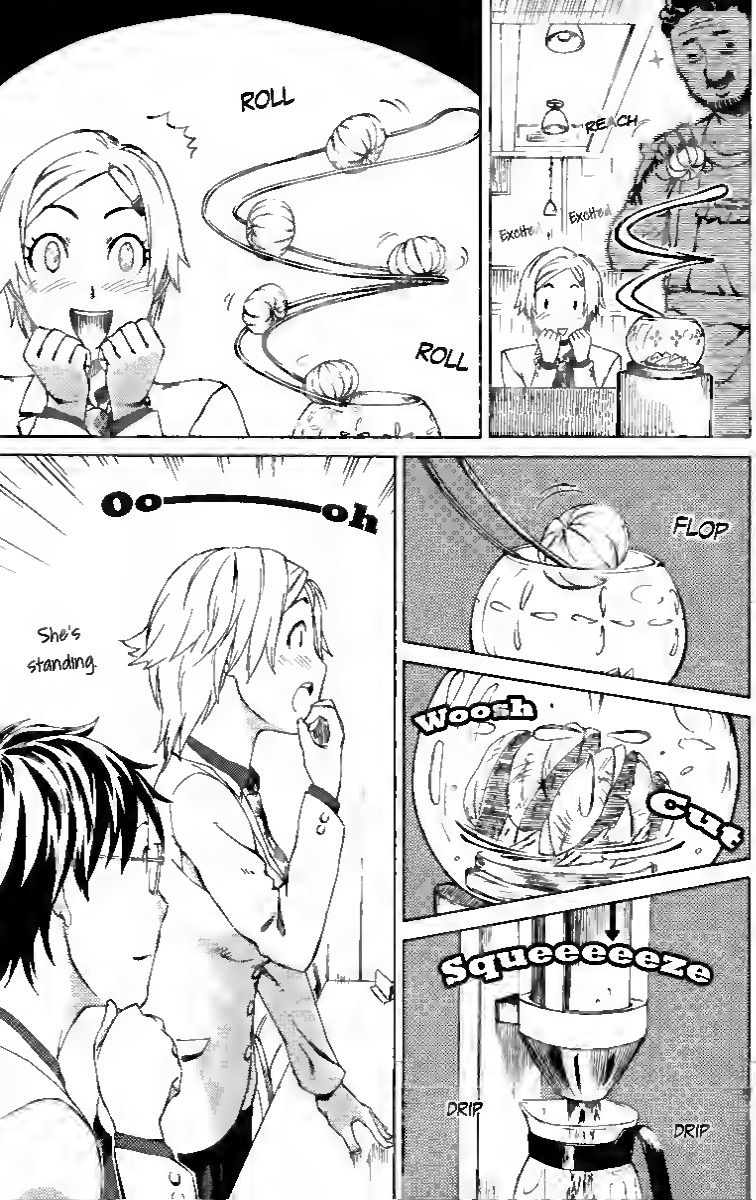

cafe
Bean'S


CLANG
CLANG
OH!

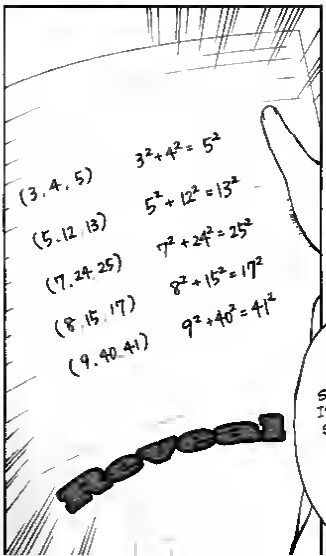

Bean's café is a favourite place of ours located in front of the station.
cafe
Bean
COFFEE
BLACK TEA
WITH CAKE
CLUBHOUSE SANDWICH
¥300
¥350
¥600
¥450
etc
CLOSE
17 00
Sometimes we talk about trifling things or about mathematics here

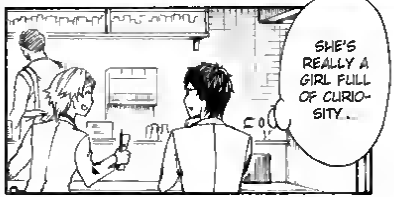
SENPAI, WHAT'S THAT?

TL NOTE: THIS IS A REFERENCE TO PYTHAGORASWITCH, A JAPANESE EDUCATIONAL PROGRAM FOR KIDS IN THIS PROGRAM, THEY SHOW RUBE GOLDBERG MACHINES CALLED "PYTHAGORA / PYTHAGOREAN DEVICES"


ROLL
ROLL
REACH
Excite
Excite
Oo——oh
She's standing
FLOP
Woosh
Cut
Squeeeeeeze
DRIP
DRIP

Kya
ah
Shine
SENPAI! THIS IS REALLY DELI-CIOUS!!


(3, 4, 5)
$3^2+4^2=5^2$
(5, 12, 13)
$5^2+12^2=13^2$
(7, 24, 25)
$7^2+24^2=25^2$
(8, 15, 17)
$8^2+15^2=17^2$
(9, 40, 41)
$9^2+40^2=41^2$
REVEAL


SHE'S REALLY A GIRL FULL OF CURIO-SITY..

Ah
SPEAKING OF PYTHAG-ORA!
RUSTLE
SINCE LAST TIME, I'VE DISCOVERED SOME PRIMITIVE PYTHAGOREAN TRIPLES!
RUSTLE

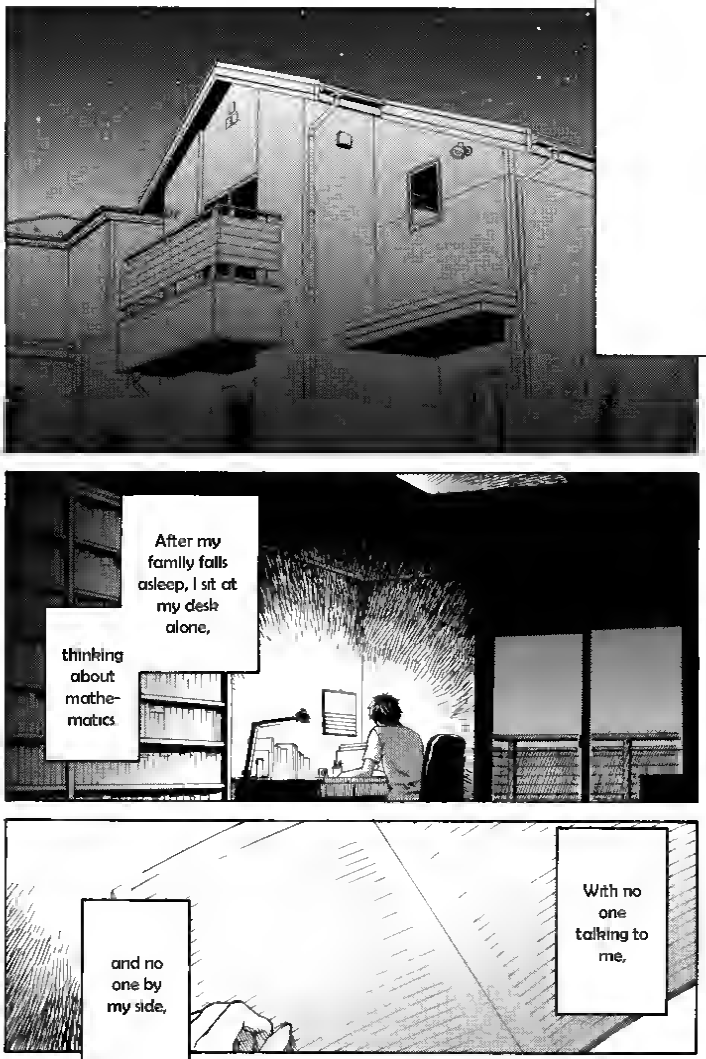
HOW DID YOU SEARCH?
Uhuh
FOLLOWING $a^2+b^2=c^2$,
I WENT BY INCREASING a IN ORDER.
AND THEN TRIED INSERTING SUITABLE NATURAL NUMBERS FOR b AND c AND SEARCHED.
I NOTICED THAT WHEN $(a, b, c) = (3, 4, 5)$ IT'S TRUE THAT $c-b = 5-4 = 1$.
FOR 4 OUT OF THE 5 PRIMITIVE PYTHAGOREAN TRIPLES,
$c-b = 1$ IS TRUE
THIS MUST BE A CLUE ABOUT SOMETHING!
CLENCH

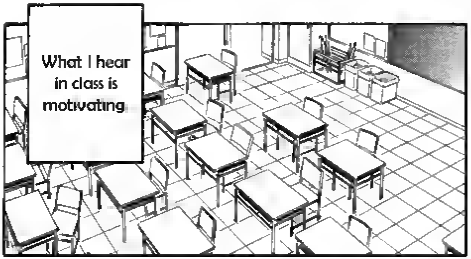
...BUT ISN'T THAT BECAUSE YOUR METHOD OF SEARCHING IS BIASED?
EH?
HAVING A SMALL MEANS
THAT WE GET A RIGHT ANGLED TRIANGLE WITH ONLY THAT SIDE SHORT.

FOR EXAMPLE WE MAY GET A LONG NARROW TRIANGLE LIKE (9, 40, 41).
SINCE IT'S LONG AND NARROW, I THINK IT'S ONLY NATURAL THAT THE LENGTHS OF THE HYPOTENUSE AND THE OTHER SIDE WILL BE CLOSE .

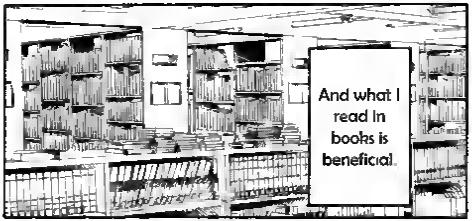
FEELING DOWN
IS. .
THAT SO?

IF YOU ADD ORANGES FROM ABOVE,
PRIMITIVE PYTHAGOREAN TRIPLES COME FROM BELOW. IF THERE WAS SUCH A MACHINE, IT WOULD BE GREAT.
JUST LIKE THAT PYTHAGORA JUICE MAKER...
A MACHINE WHERE DIFFERENT PRIMITIVE PYTHAGOREAN TRIPLES COME OUT WHEN DIFFERENT ORANGES GO IN.
ISN'T THAT RIGHT?
I DON'T UNDERSTAND ANYTHING
NOTHING AT ALL!
UMMM...
Is that so?
NOW THAT I REMEMBER, WHAT WAS IT THAT YURI WANTED TO TELL YOU IN THE HOSPITAL?

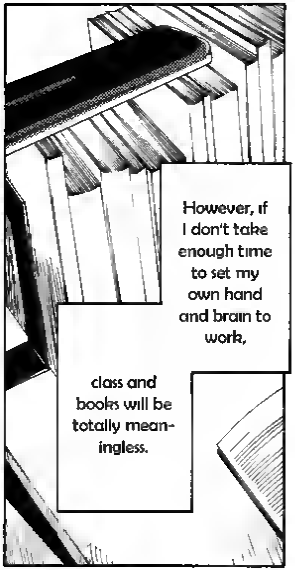

After my family falls asleep, I sit at my desk alone,
thinking about mathematics
With no one talking to me,
and no one by my side,


What I hear in class is motivating.

this is my precious time.


And what I read in books is beneficial.

Let's think carefully about Tetra-chan's problem today.


However, if I don't take enough time to set my own hand and brain to work,
class and books will be totally meaningless.

Based on Tetra-chan's problem

"Are there infinitely many primitive Pythagorean triples?"

a	b	c
(3)	4	(5)
(5)	12	(13)
(7)	24	(25)
8	(15)	(17)
(9)	40	(41)

(O is an odd number)

By looking at the table of primitive Pythagorean triples it seems that a and b cannot both be even or odd

↓ Suppose they are

«Assumptions»

- a and b are both even

Since $a^2 + b^2 = c^2$, c^2 will also be even. So will c. Then a, b, c have 2 as a common factor, which is not consistent with the definition of «primitive Pythagorean triples» requiring the HCF to be 1 → **«a and b are both even» is not correct**

- a and b are both odd

Using natural numbers J, K, we can write $a = 2J-1$, $b = 2K-1$

Let's try substituting these expressions into the Pythagorean Theorem

$$(2J-1)^2 + (2K-1)^2 = c^2$$

$$4(J^2 - J + K^2 - K) + 2 = c^2 \quad \leftarrow \text{expand and take the 4 out}$$

Since there is a tail of +2 left at the end, the left-hand side isn't divisible by 4. On the other side, since a and b are odd, $a^2 + b^2 = c^2$ is even, and moreover a multiple of 4. In other words, the right-hand side is divisible by 4. Since the left-hand side isn't divisible by 4, this is contradictory.

Neither assumption can be true.

→ **«a and b are both odd» is not correct.**

Without loss of generality, a is odd, b is even and c is odd.

Now, what shall I do...

(!) Just like what Miruka-san said: «From its prime factors, the structure of an integer is known». Decompose $a^2 + b^2 = c^2$ into prime factors...?

Anyway, let's try putting it into «product form»

$a^2 + b^2 = c^2$ ◂ Pythagorean Theorem

$b^2 = c^2 - a^2$ ◂ Move a^2 and form a «difference of two squares»

$b^2 = (c + a)(c - a)$ ◂ «The difference of two squares is the product of the sum and difference»

↓

↓

Since a is odd, b is even, c is odd

c − a, b, c + a are all even numbers

c − a = 2A, b = 2B, c + a = 2C (A, B, C natural numbers)

Let's try to write the equation $b^2 = (c + a)(c - a)$ in «product form» using A, B, C

$(2B)^2 = (2C)(2A)$ ◂ Substitute in A, B, C

$B^2 = AC$ ◂ Simplify

① Are the 2 numbers a and c «coprime»?

Assuming a and c have a highest common factor $g \geq 2$,

natural numbers g, J, K exist, such that $a = gJ$, $c = gK$.

And from the Pythagorean theorem, we can say that $b^2 = c^2 - a^2$,

$b^2 = c^2 - a^2 = (gK)^2 - (gJ)^2 = g^2(K^2 - J^2)$

b^2 will be a multiple of g^2, and so b will be a multiple of g, which means a, b, c will all be multiples of $g \geq 2$.

But this will contradict the condition that the 3 numbers a, b, c must be coprime.

Therefore, by reductionem ad absurdum, → **the 2 numbers a and c are «coprime».**

② Then, are A and C «coprime»?

Assuming that «A and C are not coprime»,

let the highest common factor of A and C be the natural number $d \geq 2$.

Then natural numbers exist such that $A = dA'$, $C = dC'$

Let's try expressing a and c using A' and C'

$(c + a) + (c - a) = 2C + 2A$ ◂ Sum up so a disappears

$c = C + A$ ◂ Simplify both sides

$c = d(C' + A')$ ◂ Express A, C using A', C'

Similarly, by evaluating $(c + a) - (c - a)$, we get $a = d(C' - A')$.

From $c = d(C' + A')$ and $a = d(C' - A')$, we can say that «a and c are both multiples of d»

In other words, $d \geq 2$ is a common factor of a and c

Which contradicts the premise that a and c are «coprime»

Therefore, by reductionem ad absurdum → **A and C are «coprime».**

... WHEN WE HAVE NATURAL NUMBERS A, B, C AND A AND C ARE COPRIME,
AND WHEN $B^2=AC$ HOLDS
IS THERE SOMETHING INTERESTING GOING ON?
"ARE THERE INFINITELY MANY PRIMITIVE PYTHAGOREAN TRIPLES?"
I DRIFTED PRETTY FAR AWAY FROM THIS ORIGINAL QUESTION...
SCRATCH
SCRATCH
SCRATCH
WHAT COULD SOMETHING INTERESTING BE?

NOW THAT I REMEMBER, MIRUKA-SAN'S SONG...
FROM ITS PRIME FACTORS, THE STRUCTURE OF AN INTEGER IS KNOWN
WHEN A, B, C ARE DECOMPOSED INTO THEIR PRIME FACTORS...
...I SEE.

When we have natural numbers A, B, C, and A and C are coprime, and when $B^2 = AC$ holds,
When A, B, C are decomposed into their prime factors, we get this.

$$A = a_1 a_2 \cdots a_s,\ B = b_1 b_2 \cdots b_t,\ C = c_1 c_2 \cdots c_u$$

(Right-hand sides are the prime factor decompositions)

- Substitute into the equation $B = AC$ and try some calculations

$(b_1 b_2 \cdots b_t)^2 = (a_1 a_2 \cdots a_s)(c_1 c_2 \cdots c_u)$ ◀ Decomposing A, B, C into prime factors
$b_1^2 b_2^2 \cdots b_t^2 = (a_1 a_2 \cdots a_s)(c_1 c_2 \cdots c_u)$ ◀ Expanding the left-hand side

When B^2 is decomposed into its prime factors, all the prime numbers b_k will be in square form b_k^2 ↓

When a square number is decomposed into its prime factors, the order of each prime factor will be an even number.

mem-o

(By the uniqueness of the prime factor decomposition, the sequences of prime factors on each side of $B^2=AC$ will perfectly coincide. Therefore, all prime factors appearing on the left must appear somewhere on the right.)

(!) Since each prime factor has even order, if A, B, C are natural numbers, $B^2=AC$ holds, and A and C are coprime, **then A and C will be square numbers**. And so A, C can be written like this

$C = m^2$, $A = n^2$ ◀ natural numbers m, n

Since there are no common prime factors between the coprime A, C, m, n will also naturally be coprime.
It should be possible to express a, b, c using the coprime m,n.

↓

- First is a.
  Since $a = C - A$, we get the following equation.

$$a = C - A = m^2 - n^2$$

  Here, since $a>0$, m will be greater than n. Also, for a to be an odd number, m and n shouldn't both be either even or odd

- Next is c.
  Since $c = C + A$, we get the following equation:

$$c = C + A = m^2 + n^2$$

- And now b.
  Since $b = 2B$... using $B^2 = AC$:

$B^2 = (n^2)(m^2)$	Since $A = n^2$, $C = m^2$
$B^2 = (mn)^2$	Rearrange
$B = mn$	Because $B>0$, $mn>0$

From here we get $b = 2B = 2mn$. In other words,

The primitive Pythagorean triples a, b, c can be expressed using pairs of coprime numbers m, n

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$$

Conversely, it can be ascertained through calculations that the triple $(m^2 - n^2, 2mn, m^2 + n^2)$ always gives primitive Pythagorean triples, where the numbers are pairwise coprime, without fail.

(!) As a result of considering parity, and decomposing to prime factors while paying attention to the condition called coprimality, I managed to obtain the general form of the primitive Pythagorean triples

«The general form of the primitive Pythagorean triples»

The primitive Pythagorean triples (a, b, c) can all be written in the following form.
(a, b are interchangeable)

$$\begin{cases} a = m^2 - n^2 \\ b = 2mn \\ c = m^2 + n^2 \end{cases}$$

- m, n are coprime
- $m < n$
- One of m, n is even, and the other is odd

IF WE GO ON CHANGING m LIKE THIS: 3, 5, 7, 11, 13...
WE WILL BE ABLE TO MAKE DIFFERENT TRIPLES (a, b, c) FROM THE PAIR m, n
AND SINCE THERE ARE INFINITELY MANY PRIME NUMBERS,
WE CAN CREATE INFINITELY MANY PRIMITIVE PYTHAGOREAN TRIPLES!
SINCE DIFFERENT PRIME NUMBERS ARE COPRIME...
...FOR EXAMPLE IF n=2, AND I MAKE m A PRIME NUMBER THAT'S 3 OR MORE,
m, n WILL SATISFY THE ABOVE CONDITIONS

WE CAN SAY THAT THERE ARE INFINITELY MANY PRIMITIVE PYTHAGOREAN TRIPLES!!
THERE-FORE,
I WONDER HOW TETRA WILL REACT WHEN I TEACH HER THIS.
USING THE PAIR OF PRIME NUMBERS m, n,
m
n
THIS IS HOW WE MAKE THE MIX JUICE CALLED THE PRIMITIVE PYTHAGOREAN TRIPLES (a, b, c), SO TO SPEAK.
a b c
Ahem

EEEH!! SOMETHING LIKE THAT!
I WOULD HAVE NEVER THOUGHT OF IT!!
I'M LOOKING FORWARD TO TOMORROW.
Grin
——MAYBE THAT'S WHAT SHE WILL SAY.
HNNN
NOW THEN.
TH..IT'S THE END OF PROOF.

Yuuki Hiroshi's extra lectures 3

The primitive Pythagorean triples.

The sets of three natural numbers where the relation $a^2+b^2=c^2$ holds like (a, b, c)= (3, 4, 5) or (a, b, c)= (6, 8, 10) are called "Pythagorean triples".

When in addition the highest common factor of the three numbers equals **1**, we call them primitive Pythagorean triples. (3, 4, 5) is a primitive Pythagorean triple, while (6, 8, 10) is not. That's because the highest common factor of 6, 8, 10 is 2 rather than 1.

To investigate the question "Are there infinitely many primitive Pythagorean triples"?, Tetra-chan has been finding examples of primitive Pythagorean triples by herself. That's a very important thing. Tetra-chan, don't get down and keep doing your best!